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On the Silov Boundary of a Pseudoconvex Domain in $\mathbb{C}^n$ with $C^{2+\alpha}$ Boundary

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Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$ and $\partial_{\text{spc}} D$ be the totality of strictly pseudoconvex boundary points. When D has a $C^{2+\alpha}$ plurisubharmonic defining function, a holomorphic diffusion process which never approaches $\partial D \backslash \overline{\partial_{\text{spc}} D}$ is constructed. This diffusion process is used to show that the Silov boundary of D coincides with $\overline{\partial_{\text{spc}} D}$.

1. INTRODUCTION

Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$. The Silov boundary $S(D)$ of D is the smallest closed subset of the boundary ∂D where the maximum principle holds for all functions h holomorphic on D and continuous up to the boundary: $\sup_{z \in D} |h(z)| = \sup_{z \in S(D)} |h(z)|$. Let $\partial_{\text{spc}} D$ be the totality of strictly pseudoconvex boundary points. For the definition, see the end of Section 2. When D has a C^3 plurisubharmonic (abbreviated psh) defining function, Debiard and Gaveau [1] have shown that $S(D)$ coincides with the closure $\overline{\partial_{\text{spc}} D}$ by constructing a suitable Kähler diffusion process. In this paper, employing another holomorphic diffusion process, we see that $S(D) = \overline{\partial_{\text{spc}} D}$ in the case where D has a $C^{2+\alpha}$ psh defining function for some $0 < \alpha \leqslant 1$.

A holomorphic diffusion process $\mathbb{M} = (Z_t, \zeta, P_z)$ on D is by definition a $C_0^\infty(D)$-regular symmetric diffusion process with the lifetime ζ such that $h(Z_{t \wedge \tau_K})$ is a P_z-martingale for $\mathbb{M}$-q.e. $z \in D$, where h is holomorphic on D, K is a compact set in D, $\tau_K = \inf\{t > 0 : Z_t \notin K\}$, and by "$\mathbb{M}$-q.e." we mean "except for a set of zero capacity with respect to the 1-capacity associated with $\mathbb{M}$." For the definition of symmetric diffusion processes and the associated 1-capacity, see [2]. A Kähler diffusion process, the minimal diffusion process generated by $\Delta/2$, Δ being the Laplace–Beltrami operator associated with a Kähler metric, is a typical example of holomorphic diffusion processes.

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By the martingale convergence theorem, we obtain

$$P_z[Z_{\zeta-} \equiv \lim_{t\uparrow\zeta} Z_t \text{ exists}] = 1, \qquad \mathbb{M}\text{-q.e. } z \in D, \tag{1.1}$$

since D is bounded. Then, for each h holomorphic on D and continuous up to ∂D, it holds that

$$h(z) = E_z[h(Z_{\zeta-})], \qquad \mathbb{M}\text{-q.e. } z \in D,$$

where E_z stands for the expectation with respect to P_z. Suppose that

$$P_z[Z_{\zeta-} \in \partial D] = 1, \qquad \mathbb{M}\text{-q.e.} \tag{1.2}$$

Then, if we define an open subset $\Gamma(\mathbb{M})$ of ∂D by

$$\Gamma(\mathbb{M}) = \bigcup \{U : U \text{ is an open set in } \mathbb{C}^n \text{ such that } U \cap \partial D \neq \phi \\ \text{and } P_z[Z_{\zeta-} \in U \cap \partial D] = 0,\ \mathbb{M}\text{-q.e. } z \in D\}, \tag{1.3}$$

then $S(D) \subset D \backslash \Gamma(\mathbb{M})$. Thus, if $\partial D \backslash \overline{\partial_{\text{spc}} D} \subset \Gamma(\mathbb{M})$ then $S(D) = \overline{\partial_{\text{spc}} D}$, because it is well known that $\overline{\partial_{\text{spc}} D} \subset S(D)$ [9]. It is this argument that is taken advantage of by Debiard and Gaveau [1]. Indeed, they have constructed a Kähler diffusion process $\mathbb{M}_0$ such that $\partial D \backslash \overline{\partial_{\text{spc}} D} \subset \Gamma(\mathbb{M}_0)$ in the case where D has a C^3 psh defining function. Motivated by their work, the author studied the boundary behaviour of holomorphic diffusion processes in [6, 7] and its applications to the complex Monge–Ampère operator. However, it has been open whether there is a holomorphic diffusion process which does not approach $\partial D \backslash \overline{\partial_{\text{spc}} D}$ when the regularity of the boundary is less than C^3. In the present paper, our aim is to answering the question affirmatively:

THEOREM. *Assume that there exist a domain Ω, a function $\sigma \in C^2(\Omega)$, which is psh on Ω, and a real number $0 < \alpha \leqslant 1$ such that $\bar{D} \subset \Omega$, $D = \{\sigma < 0\}$, $d\sigma \neq 0$ on ∂D, and $\partial^2\sigma/\partial z^i\, \partial \bar{z}^j$, $1 \leqslant i, j \leqslant n$, are all α-Hölder continuous. Then, there exists a holomorphic diffusion process $\mathbb{M}$ such that $\partial D \backslash \overline{\partial_{\text{spc}} D} \subset \Gamma(\mathbb{M})$.*

It follows from the theorem that

COROLLARY. *Let D satisfy the same assumption as stated in the theorem. Then $S(D) = \overline{\partial_{\text{spc}} D}$.*

2. PRELIMINARIES

Let G be a bounded domain in $\mathbb{C}^n$. A C^2 function σ on G is said to be psh if the matrix $((\partial^2\sigma/\partial z^i\, \partial \bar{z}^j)(z))_{1 \leqslant i, j \leqslant n}$ is nonnegative definite at each

$z \in G$. If the matrix is positive definite at every $z \in G$, σ is called strictly psh. We establish that:

LEMMA 2.1. *Let σ, $\tau \in C^2(G)$ be both negative and psh on G. Then, $-(-\sigma)^\delta (-\tau)^\varepsilon$ is psh for any $\delta > 0$ and $\varepsilon > 0$ with $\delta + \varepsilon \leqslant 1$. Moreover, $-(-\sigma)^\delta (-\tau)^\varepsilon$ is strictly psh if either σ or τ is so.*

Proof. For the sake of simplicity, for a C^2 function f, we write $\partial\bar\partial f$, and ∂f, $\bar\partial f$ for the matrix $(\partial^2 f/\partial z^i \partial \bar z^j)_{1 \leqslant i, j \leqslant n}$ and the vectors $(\partial f/\partial z^i)_{1 \leqslant i \leqslant n}$, $(\partial f/\partial \bar z^i)_{1 \leqslant i \leqslant n}$, respectively. For $\xi = (\xi^1, ..., \xi^n)$, $\eta = (\eta^1, ..., \eta^n) \in \mathbb{C}^n$, $\xi \cdot \eta$ denotes the matrix $(\xi^i \eta^j)_{1 \leqslant i, j \leqslant n}$.

Let $p = -(-\sigma)^\delta (-\tau)^\varepsilon$. Since

$$\partial p = (-p)\left(\frac{\delta}{-\sigma}\,\partial\sigma + \frac{\varepsilon}{-\tau}\,\partial\tau\right),$$

by differentiating $-\log(-p)$, we obtain

$$\begin{aligned}\frac{1}{-p}\,\partial\bar\partial p = \frac{\varepsilon}{-\sigma}\,\partial\bar\partial\sigma + \frac{\varepsilon}{-\tau}\,\partial\bar\partial\tau + \frac{\delta-\delta^2}{(-\sigma)^2}\,\partial\sigma\cdot\bar\partial\sigma + \frac{\varepsilon-\varepsilon^2}{(-\tau)^2}\,\partial\tau\cdot\bar\partial\tau \\ -\frac{\delta\varepsilon}{\sigma\tau}\,(\partial\sigma\cdot\bar\partial\tau + \partial\tau\cdot\bar\partial\sigma). \qquad (2.1)\end{aligned}$$

Let us denote by $\langle\ ,\ \rangle$ the standard inner product in $\mathbb{C}^n$ and set $L_f(z:\xi) = \sum_{i,j=1}^n \xi^i(\partial^2 f/\partial z^i \partial \bar z^j)(z)\,\bar\xi^j$ for $f \in C^2(G)$ and $\xi = (\xi^1, ..., \xi^n) \in \mathbb{C}^n$. Then, (2.1) yields

$$\begin{aligned}\frac{1}{-p}\,L_p(z:\xi) \geqslant \frac{\delta}{-\sigma}\,L_\sigma(z:\xi) + \frac{\varepsilon}{-\tau}\,L_\tau(z:\xi) \\ + (\delta-\delta^2)\left\{\left|\left\langle \xi, \frac{1}{-\sigma}\,\bar\partial\sigma(z)\right\rangle\right| - \frac{\varepsilon}{1-\delta}\left|\left\langle \xi, \frac{1}{-\tau}\,\bar\partial\tau(z)\right\rangle\right|\right\}^2 \\ + \frac{\varepsilon(1-\delta-\varepsilon)}{1-\delta}\left|\left\langle \xi, \frac{1}{-\tau}\,\bar\partial\tau(z)\right\rangle\right|^2.\end{aligned}$$

The assertion from this inequality follows immediately.

The identity (2.1) also implies the following:

LEMMA 2.2. *Let σ, $\tau \in C^2(D)$ be both negative and psh on G. Suppose that $-1 \leqslant \sigma$ and that*

$$M \equiv \sup\left\{|\tau(z)|,\ \left|\frac{1}{\tau(z)}\right|,\ |\partial\tau(z)|,\ \|\partial\bar\partial\tau(z)\| : z \in D\right\} < +\infty,$$

where for a $\mathbb{C}$*-valued* $n \times n$*-matrix* A, $\|A\|^2 = \operatorname{trace} AA^*$. *Then, it holds that*

$$\text{the minimal eigenvalue of } \partial\bar\partial(-(-\sigma)^\delta(-\tau)^\varepsilon)(z)$$
$$\leqslant M^\varepsilon\{\delta + \varepsilon(M^2 + M^3)\}(-\sigma)^{\delta-1}\{L_\sigma(z:\xi) + (-\sigma(z))\}$$

for every $z \in G$ *and* $\xi \in \mathbb{C}^n$ *with* $|\xi| = 1$ *and* $\langle \xi, \bar\partial\sigma(z)\rangle = 0$.

We end this section by recalling the definition of strictly pseudoconvex points. A domain G is said to possess a C^2 psh defining function if there are an open set O in $\mathbb{C}^n$ and $\psi \in C^2(O)$ such that $\bar G \subset O$, $G \cap O = \{\psi < 0\}$ and $d\psi \neq 0$ on ∂G. Such a function ψ is called a C^2 psh defining function of G. For a bounded domain G possessing a C^2 psh defining function ψ, we set

$$\partial_{\mathrm{spc}} G = \{z \in \partial G : L_\psi(z:\xi) > 0 \text{ for every } \xi \in \mathbb{C}\backslash\{0\} \text{ with } \langle \xi, \bar\partial\psi(z)\rangle = 0\}.$$

3. Proof of Theorem

We first fix the notation which we will use. Without loss of generality we may assume that

$$-1 \leqslant \sigma(z) < 0, \qquad z \in D. \tag{3.1}$$

Take $0 < \beta$, $\gamma < 1$, and $R > 0$ such that

$$\alpha + (\beta\gamma - 1)(n-1) > 0 \tag{3.2}$$

$$R > 2\sup\{|z|^2 : z \in D\}. \tag{3.3}$$

We set

$$p = -(-\sigma)^\beta (R - |z|^2)^{1-\beta}. \tag{3.4}$$

By Lemma 2.1, p and $-(-p)^\gamma$ are both strictly psh on D. Define a positive current θ of bidegree $(n-1, n-1)$ by

$$\theta(dz^i \wedge \sqrt{-1}\, d\bar z^j) = a^{ij}\, dV, \qquad 1 \leqslant i, j \leqslant n, \tag{3.5}$$

where (a^{ij}) is the cofactor matrix of $\partial\bar\partial(-(-p)^\gamma)$ and V is the Lebesgue measure on D. Let m be an everywhere dense positive Radon measure on D given by

$$dm = (-\sigma)^{\alpha + (\beta\gamma - 1)(n-1) - 2}\, dV. \tag{3.6}$$

We define a symmetric form $\mathscr{E}_\theta$ by

$$\mathscr{E}_\theta(f, g) = \int_D df \wedge d^c g \wedge \theta, \qquad f, g \in C_0^\infty(D).$$

Note that

$$\theta = \frac{1}{2^{2n-1}(n-1)!}\{dd^c(-(-p)^\gamma)\}^{n-1} \tag{3.7}$$

and hence θ is closed. It follows from (3.5), (3.6), and the strict plurisubharmonicity of $-(-p)^\gamma$ that for every relatively compact open $G \subset D$,

$$dd^c |z|^2 \wedge \theta \leqslant C_G m \qquad \text{on } G$$

for some $C_G > 0$. Thus, by an argument similar to that in [4, Lemma 1], we see that $\mathscr{E}_\theta$ is closable on $L^2(D : m)$. The closure $\mathscr{E}$ is a $C_0^\infty(D)$-regular, local, Dirichlet form on $L^2(D : m)$ such that every holomorphic function h on D is $\mathscr{E}$-harmonic. $\mathscr{E}$ then admits a unique holomorphic diffusion process $\mathbb{M} = (Z_t, \zeta, P_z)$ up to equivalence. See [4, 8]. We now show

LEMMA 3.1. *$\mathbb{M}$ explodes*:

$$P_z[\zeta < +\infty] = 1, \qquad \mathbb{M}\text{-}q.e.\ z \in D.$$

In particular, (1.2) *is satisfied.*

Proof. Let $(\mathscr{F}, \mathscr{E})$ be the Dirichlet space of $\mathbb{M}$ and $(\mathscr{F}', \mathscr{E}')$ be that of the absorbing barrier Brownian motion on D. Then, for every relatively compact open set G in D, there is a constant $C_0 > 0$ such that

$$\mathscr{E}(f, f) \geqslant C_0 \mathscr{E}'(f, f), \qquad f \in C_0^\infty(G)$$

and

$$C_0 V \leqslant m \leqslant C_0^{-1} V.$$

By [3, Proposition 1.5], we see that $\mathbb{M}$ is irreducible. Then, as is seen in [7, 10], $\mathbb{M}$ is transient.

We now recall that if $D_0 = \{\varphi < 0\}$ is a bounded pseudoconvex domain in $\mathbb{C}^n$ with a continuous psh defining function φ and $\mathbb{N}$ is a transient holomorphic diffusion process with the symmetrising measure m_0, then $\mathbb{N}$ explodes whenever $\int_{D_0} |\varphi|\, dm_0 < \infty$. See [10, Theorem 1.1]. Because of (3.2), it is elementary to see that

$$\int_D |\sigma|\, dm < +\infty.$$

Thus, we obtain that $\mathbb{M}$ explodes.

To see that $\partial D \backslash \overline{\partial_{\mathrm{spc}} D} \subset \Gamma(\mathbb{M})$, we fix an arbitrary $z^* \in \partial D \backslash \overline{\partial_{\mathrm{spc}} D}$ and an open set U such that $U \cap \partial D \subset \partial D \backslash \overline{\partial_{\mathrm{spc}} D}$. Shrinking U if necessary, we may assume that a mapping $\Phi : (\partial D \cap U) \times (-\delta_0, \delta_0) \to U$ given by

$$\Phi(w, a) = w + a n_w,$$

n_w being the outer normal vector at w, is a C^2-diffeomorphism for some $\delta_0 > 0$. For $z \in U$, $w \in \partial D$ satisfying $\Phi(w, a) = z$ is denoted by $\pi(z)$. Shrinking U again if necessary, we may, moreover, assume that there are C_1, $C_2 > 0$ such that $C_1 \delta \leqslant \frac{1}{2}$ and

$$\partial\sigma(z) \neq 0 \tag{3.8}$$

$$|\partial\sigma(z)|^{-1}\, |\bar{\partial}\sigma(z) - 4\langle \bar{\partial}\sigma(\pi(z)), \bar{\partial}\sigma(z)\rangle\, \bar{\partial}\sigma(\pi(z))| \leqslant C_1\, |z - \pi(z)| \tag{3.9}$$

$$|z - \pi(z)| \leqslant C_2\, |\sigma(z)| \tag{3.10}$$

for every $z \in U$. On such a U, we obtain the following inequality, which is verified by modifying the argument used in [1].

LEMMA 3.2. *Let U be as above. Then, for some $C_3 > 0$, it holds that*

$$C_3\, dd^c |z|^2 \wedge \theta \geqslant (-\sigma)^{-\beta\gamma - \alpha + 1}\, dd^c(-(-p)^{\gamma}) \wedge \theta \qquad \text{on} \quad U \cap D. \tag{3.11}$$

Proof. Take an arbitrary $z \in U \cap D$ and fix it. Choose $\xi \in \mathbb{C}^n$ such that $|\xi| = 1$, $\langle \xi, \bar{\partial}\sigma(\pi(z))\rangle = 0$, and $L_\sigma(\pi(z) : \xi) = 0$. We put

$$\eta = \xi - |\bar{\partial}\sigma(z)|^{-2}\, \langle \xi, \bar{\partial}\sigma(z)\rangle\, \bar{\partial}\sigma(z).$$

Since $\langle \xi, \bar{\partial}\sigma(\pi(z))\rangle = 0$ and $|\bar{\partial}\sigma(\pi(z))| = \frac{1}{2}$, it follows from (3.9) that

$$\begin{aligned} |\eta - \xi| &\leqslant |\bar{\partial}\sigma(z)|^{-1}\, |\bar{\partial}\sigma(z) - 4\langle \bar{\partial}\sigma(z), \bar{\partial}\sigma(\pi(z))\rangle\, \bar{\partial}\sigma(\pi(z))| \\ &\leqslant C_1\, |z - \pi(z)| \\ &\leqslant C_1 \delta \leqslant \tfrac{1}{2}. \end{aligned} \tag{3.12}$$

In particular,

$$\tfrac{1}{2} \leqslant |\eta| \leqslant \tfrac{3}{2}. \tag{3.13}$$

Define constants C_4 and C_5 by

$$C_4 = \sup_{u, v \in U} \frac{\|\partial\bar{\partial}\sigma(u) - \partial\bar{\partial}\sigma(v)\|}{|u - v|^{\alpha}} < \infty$$

$$C_5 = \sup_{w \in U} \|\partial\bar{\partial}\sigma(w)\| < \infty.$$

Then, by a straightforward computation, we obtain that

$$\begin{aligned} L_\sigma(z:\eta) &= L_\sigma(z:\eta) - L_\sigma(\pi(z):\xi) \\ &\leqslant (|\eta|+1)\, C_5\, |\eta-\xi| + C_4\, |z-\pi(z)|^\alpha. \end{aligned}$$

Combining this with (3.10), (3.12), and (3.13), we have

$$L_\sigma\left(z:\frac{\eta}{|\eta|}\right) \leqslant 2(5C_1C_5 + 2C_4)\, C_2\, |\sigma(z)|^\alpha. \tag{3.14}$$

Because of Lemma 2.2, (3.14) implies the existence of a constant $C_6 > 0$, independent of z, such that

$$\text{the minimal eigenvalue of } \partial\bar{\partial}(-(-p)^\gamma)(z) \leqslant C_6(-\sigma(z))^{\beta\gamma+\alpha-1}. \tag{3.15}$$

It follows from (3.5) and (3.7) that

$$dd^c\, |z| \wedge \theta = 2\,\text{trace}(a^{ij})\, dV, \tag{3.16}$$

$$dd^c(-(-p)^\gamma) \wedge \theta = 2n\,\det(\partial\bar{\partial}(-(-p)^\gamma))\, dV. \tag{3.17}$$

If we set $(b^{ij}) = (\partial\bar{\partial}(-(-p)^\gamma))^{-1}$, then (3.15) implies that

$$\begin{aligned} \text{trace}(a^{ij}) &= \text{trace}(b^{ij})\det(\partial\bar{\partial}(-(-p)^\gamma)) \\ &\geqslant C_6^{-1}(-\sigma)^{-\beta\gamma-\alpha+1}\det(\partial\bar{\partial}(-(-p)^\gamma)) \qquad \text{on} \quad U\cap D. \end{aligned}$$

Combining this with (3.16), (3.17), we obtain the desired inequality.

We now consider the relationship between the measures m and $dd^c(-\log(-p)) \wedge \theta$. We verify

LEMMA 3.3. *Let U be as before. Then, there is a constant C_7 such that*

$$dd^c(-\log(-p)) \wedge \theta \leqslant C_7 m \qquad \text{on} \quad U\cap D. \tag{3.18}$$

Proof. It is easy to show that

$$dd^c(-\log(-p)) \wedge \theta \leqslant \gamma(1-\gamma)\, R^{-\beta\gamma}(-\sigma)^{-\beta\gamma}\, dd^c(-(-p)^\gamma) \wedge \theta \qquad \text{on } D. \tag{3.19}$$

Plugging (3.11) into (3.19), we obtain that

$$dd^c(-\log(-p)) \wedge \theta \leqslant \gamma(1-\gamma)\, R^{-\beta\gamma}\, C_3(-\sigma)^{\alpha-1}\, dd^c\, |z|^2 \wedge \theta \qquad \text{on } U\cap D. \tag{3.20}$$

Take an arbitrary $z \in U \cap D$ and fix it. Choose $\xi_1, ..., \xi_n \in \mathbb{C}^n$ such that $\langle \xi_i, \xi_j \rangle = \delta_{ij}$ and $\langle \xi_j, \bar{\partial}\sigma(z) \rangle = 0$, $1 \leqslant j \leqslant n-1$. It follows from (2.1) that there exists a constant C_8, independent of z, such that

$$|\langle \xi_i, \bar{\partial}\partial(-(-p)^\gamma)(z)\, \xi_j \rangle| \leqslant C_8(-\sigma(z))^{\beta\gamma - 1} \qquad \text{if} \quad i+j<2n \tag{3.21}$$

and

$$|\langle \xi_n, \bar{\partial}\partial(-(-p)^\gamma)(z)\, \xi_n \rangle| \leqslant C_8(-\sigma(z))^{\beta\gamma - 2}. \tag{3.22}$$

Let (c^{ij}) be the cofactor matrix of $(\langle \xi_i, \bar{\partial}\partial(-(-p)^\gamma)(z)\, \xi_j \rangle)_{1 \leqslant i,j \leqslant n}$. Note that

$$\operatorname{trace}(a^{ij}) = \operatorname{trace}(c^{ij}).$$

Hence, (3.16), (3.21), and (3.22) imply that

$$dd^c\, |z|^2 \wedge \theta \leqslant C_9(-\sigma)^{(\beta\gamma - 1)(n-1)-1}\, dV \qquad \text{on} \quad U \cap D$$

for some constant C_9. Combining this with (3.20), we obtain that (3.18) holds.

We finally investigate the behaviour of $\mathbb{M}$ on $U \cap D$ in the following lemma. The assertion of the theorem is an immediate consequence of the lemma, because of the maximality of $\Gamma(\mathbb{M})$.

LEMMA 3.4. *Let U be as above. Then, it holds that*

$$P_z[Z_{\zeta-} \in \partial M \cap U] = 0 \qquad \mathbb{M}\text{-}q.e.\ z \in D.$$

Proof. Recall that $-\log(-p)$ is locally in $\mathscr{F}$, where $\mathscr{F}$ is the domain of the Dirichlet form associated with $\mathbb{M}$. See [4, 5]. The continuous additive functional (abbreviated CAF) $-\log(-p(Z_t)) + \log(-p(Z_0))$ is decomposed as

$$-\log(-p(Z_t)) + \log(-p(Z_0)) = M_t + N_t, \qquad t < \zeta,$$

where M_t is a local martingale CAF and N_t is a CAF of local energy zero (cf. [2]). By the standard time change argument, we have that

$$-\log(-p(Z_t)) + \log(-p(Z_0)) = B(\langle M_t \rangle) + N_t, \qquad t < \zeta, \tag{3.23}$$

where $\langle M \rangle$ is the quadratic variation process of M and $B(t)$ is an $\mathbb{R}^1$-valued Brownian motion with $B(0) = 0$.

The Revuz measures of N and $\langle M \rangle$ are $dd^c(-\log(-p)) \wedge \theta$ and $d(-\log(-p)) \wedge d^c(-\log(-p)) \wedge \theta$, respectively. See [4, 6]. Since the latter measure dominates the former one, it follows from (3.18) that

$$\langle M \rangle_t \leqslant N_t \leqslant C_7 t, \qquad t < \zeta \wedge \tau', \tag{3.24}$$

under P_z, $\mathbb{M}$-q.e. $z \in U \cap D$, where $\tau' = \inf\{t > 0 : Z_t \in D \backslash U\}$. Equation (3.24) and Lemma 3.1 yield that

$$P_z[\lim_{t \uparrow \zeta \wedge \tau'} \langle M \rangle_t < \infty, \lim_{t \uparrow \zeta \wedge \tau'} N_t < \infty] = 1, \qquad \mathbb{M}\text{-q.e. } z \in U \cap D.$$

Combined with (3.23), this implies that

$$P_z[Z_t \in D \cap U, \ t < \zeta] = 0, \qquad \mathbb{M}\text{-q.e. } z \in U \cap D.$$

Hence there exists a Borel set $E \subset D$ such that E is of zero capacity with respect to the 1-capacity associated with $\mathbb{M}$,

$$P_z[Z_t \in D \backslash E] = 1, \qquad z \in D \backslash E, \tag{3.25}$$

$$P_z[Z_t \in D \cap U, \ t < \zeta] = 0, \qquad z \in U \backslash E. \tag{3.26}$$

Set

$$A_r = \{\zeta > r \text{ and } Z_t \in D \cap U \text{ for } r \leqslant t < \zeta\}.$$

By the Markov property, (3.25), and (3.26), we obtain that

$$P_z[A_r] = 0 \qquad \text{for} \quad z \in D \backslash E,$$

Since $\{Z_{\zeta-} \in \partial D \cap U\} \subset \bigcup_r A_r$, where the union is taken over all non-negative rational numbers, this implies that

$$P_z[Z_{\zeta-} \in \partial D \cap U] = 0, \qquad \mathbb{M}\text{-q.e. } z \in D.$$

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